Math 564: Advance Analysis 1
Lecture 7

Borel measures. For a topological space X, a Borel measure is just a measure defined on the Borel σ-algebra $\mathcal{B}(X)$ of X.

Regularity. Let X be a metric space and let μ be a finite Borel measure on X. Then μ is regular, i.e. for each μ-measurable $A \subseteq X$

$$\left.\begin{aligned} \mu(A) &= \inf\{\mu(U) : A \subseteq U \text{ open}\} \quad (*.\text{outer}) \\ &= \sup\{\mu(C) : A \supseteq C \text{ closed}\}. \quad (*.\text{inner}) \end{aligned}\right\} (*)$$

In particular, (i) μ is strongly regular, i.e. for each μ-measurable set $A \subseteq X$,

$$\begin{aligned} 0 &= \inf\{\mu(U \setminus A) : A \subseteq U \text{ open}\} \\ &= \inf\{\mu(A \setminus C) : A \supseteq C \text{ closed}\}. \end{aligned}$$

This means the symmetric difference is μ-null.

(ii) A set $A \subseteq X$ is μ-measurable $\iff$ $A \approx^{\mu}$ a G_δ set $\supseteq A$

$\iff$ $A \approx^{\mu}$ an F_σ-set $\subseteq A$.

Proof. Part (i) follows immediately from regularity due to the finiteness of measure. Part (ii) follows from part (i): for a μ-meas. A, let $U_n \supseteq A$ open s.t. $\mu(U_n \setminus A) \leq \frac{1}{n+1}$, then $D := \bigcap_{n \in \mathbb{N}} U_n$ is G_δ containing A and $\mu(D \setminus A) \leq \mu(U_n \setminus A) \leq \frac{1}{n+1}$ $\forall n$, so $\mu(D \setminus A) = 0$. The F_σ statement is proven analogously.

(*) Let $\mathcal{S}$ be the collection of all μ-measurable sets satisfying (*).

Claim (a). $\mathcal{S}$ contains all open sets.

Pf. Open sets trivially satisfy (*.outer) and in a metric space, each open set U is F_σ, i.e. $U = \bigcup_{n \in \mathbb{N}} C_n$. Replacing C_n with $\bigcup_{i \leq n} C_i$, we may assume WLOG that the C_n are increasing. Then $\mu(U) = \lim_n \mu(C_n)$. $\boxtimes$

Claim (b). $\mathfrak{S}$ is closed under complements.

Pf. Let $A \in \mathfrak{S}$ and $\varepsilon > 0$. Take $U \supseteq A$ be open and $\mu(U) \approx_\varepsilon \mu(A)$ and let $C \subseteq A$ be closed and $\mu(C) \approx_\varepsilon \mu(A)$. Then $U^c \subseteq A^c$ is closed and $\mu(U^c) = \mu(X) - \mu(U) \approx_\varepsilon \mu(X) - \mu(A) = \mu(A^c)$ by the finiteness of μ, and likewise, $C^c \supseteq A$ is open and $\mu(C^c) \approx \mu(A^c)$. ⊠

Claim (c). $\mathfrak{S}$ is closed under finite unions.

Pf. This follows from the fact that finite unions of open/closed sets are open/closed. ⊠

Claim (d). $\mathfrak{S}$ is closed under ctbl unions.

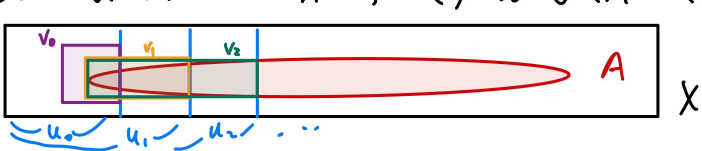
Pf. Let $A_n \in \mathfrak{S}$ aiming to show that $\bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{S}$. By Claim (c), we can replace A_n with $\bigcup_{i \leq n} A_n$ and assume WLOG that the A_n are increasing. Fix $\varepsilon > 0$ and take $C_n \subseteq A_n \subseteq U_n$ where C_n is closed, U_n is open, and $\mu(C_n) \approx_{\varepsilon/2^{n+1}} \mu(A_n) \approx \mu(U_n)$. Then $\mu(\bigcup_n U_n \setminus \bigcup_n A_n) \leq \sum_n \mu(U_n \setminus A_n) \leq \varepsilon$. For closed sets, note that $\mu(\bigcup^{\uparrow} A_n) = \lim_n \mu(A_n)$ and let N be large enough so that $\mu(A_N) \approx_{\varepsilon/2} \mu(\bigcup^{\uparrow}_n A_n)$. Then $\mu(C_N) \approx_{\varepsilon/2^{N+1}} \mu(A_N) \approx_{\varepsilon/2} \mu(\bigcup^{\uparrow}_n A_n)$. ⊠

Thus, $\mathfrak{S}$ contains all Borel sets. For a μ-measurable set A, write $A = B_0 \cup Z_0$ and $A = B_1 \setminus Z_1$, where B_0, B_1 are Borel and Z_0, Z_1 are μ-null. Then for each $\varepsilon > 0$, there is a closed $C \subseteq B_0 \subseteq A$ and open $U \supseteq B_1 \supseteq A$ s.t. $\mu(C) \approx_\varepsilon \mu(B_0) = \mu(A) = \mu(B_1) \approx_\varepsilon \mu(U)$. Hence, $\mathfrak{S} = \mathrm{Meas}_\mu$. □

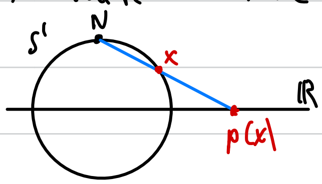
Cor. Let μ be a σ-finite Borel measure on a metric space X. If $X = \bigcup_{n \in \mathbb{N}} U_n$ where the U_n are open sets of finite measure, then μ is *strongly regular*.

Proof. **(*.outer):** Let $A \subseteq X$ be μ-meas. Then for each $n \in \mathbb{N}$, $\exists\, V_n \subseteq U_n$ open in U_n, and hence in X since U_n itself is open, such that $V_n \supseteq A \cap U_n$ and and $\mu(V_n \setminus (A \cap U_n)) < \varepsilon/2^{n+1}$. But then $\mu(\bigcup_n V_n \setminus A) = \mu(\bigcup_n V_n \setminus \bigcup_n (A \cap U_n)) \leq \sum_n \mu(V_n \setminus (A \cap U_n)) \leq \sum \varepsilon/2^{n+1} = \varepsilon$.

(*.inner): Let $A \subseteq X$ be μ-meas. By (*.outer) applied to A^c, $\exists$ open $U \supseteq A^c$ s.t. $\mu(U \setminus A^c) \leq \varepsilon$. But $U \setminus A^c = U \cap A = A \setminus U^c$, so $\mu(A \setminus U^c) \leq \varepsilon$ and U^c is closed. □



Caution. One cannot remove the condition of X being a ctbl union of open sets of finite measure. Here is a counter-example. Take X be the one-point compactification of $\mathbb{R}$, i.e. $X := \mathbb{R} \cup \{\infty\}$, where the topology is generated by the usual open subsets of $\mathbb{R}$ together with sets of the form $(-\infty, a) \cup (b, +\infty) \cup \{\infty\}$. Note that X is homeomorphic to S^1 via "wrapping $\mathbb{R}$ around S^1 minus its north pole N and mapping ∞ to N (more precisely, via the stereographic projection $p: S^1 \to X$ mapping N to ∞). Thus, X is metrizable, i.e. admits a metric (copied from S^1 via p^{-1}) generating the topology. Let λ be the Borel measure on X that coincides with the Lebesgue measure on $\mathbb{R}$ and $\lambda(\infty) := 0$. Then λ is σ-finite but every open set U containing ∞ has infinite measure because it contains a set of the form $(-\infty, a) \cup (b, +\infty)$. Thus, $\lambda(\{\infty\}) = 0 \neq \infty = \inf\{\lambda(U) : U \supseteq \{\infty\} \text{ open}\}$. $\square$



Tightness. Let μ be a finite Borel measure on a Polish space (X, d), where d is a complete. Then μ is tight: for each meas. set A,

$$\mu(A) = \sup\{\mu(K) : A \supseteq K \text{ compact}\}.$$

Proof. Firstly, we may assume A is closed by the regularity of μ. Since A is closed, the metric space (A, d) is still Polish, so we may restrict to A and assume that $A = X$. Thus we need to show that $\forall \varepsilon > 0$ $\exists$ compact $K \subseteq X$ s.t. $\mu(X \setminus K) \leq \varepsilon$.

<u>Recall:</u> A set in a complete metric space is compact iff it is closed and totally bdd.

For each n (i.e. $\frac{1}{n}$), let $(B_{n,i})_{i \in \mathbb{N}}$ be a sequence of closed balls of size $\leq \frac{1}{n}$ with $X = \bigcup_{i \in \mathbb{N}} B_{n,i}$. Such a sequence exists by separability of X.

Noting that $X = \bigcup_N \left(\bigcup_{i \le N} B_{n,i}\right)$, there is a large enough k_n s.t. $\mu\left(X \setminus \bigcup_{i \le k_n} B_{n,i}\right) < \varepsilon/2^{n+1}$. The set $C_n := \bigcup_{i \le k_n} B_{n,i}$ is closed and admits a finite $\frac{1}{n}$-net (a cover with sets of diameter $\le \frac{1}{n}$). Then $K := \bigcap_n C_n$ is closed and totally bdd and $\mu(X \setminus K) \le \mu\left(\bigcup_n C_n^c\right) \le \sum_n \mu(C_n^c) \le \sum_n \varepsilon/2^{n+1} = \varepsilon$. $\square$

99% lemmas. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and suppose that the σ-algebra $\mathcal{B}$ is generated by an algebra $\mathcal{A}$. Then we already know that every μ-meas. set M is approximated by sets from $\mathcal{A}$, in particular, $\exists A \in \mathcal{A}$ s.t. $\mu(A \Delta M) < 0.5\%$ of $\mu(M)$ and $\mu(A) \approx_\varepsilon \mu(M)$, where $\varepsilon < 0.5\% \mu(M)$, so $\mu(A \Delta M) < 1\% \mu(A)$. In particular, 99% of our nice set A is M. Thus, instead of working with M we work with A.

99% for Lebesgue and Bernoulli.

(a) For $(\mathbb{R}^d, \lambda)$, for any λ-measurable non-null set A, there is a box B whose 99% is A, i.e. $\frac{\lambda(B \cap A)}{\lambda(B)} \ge 0.99$.

(b) For $(2^{\mathbb{N}}, \mu_p)$, for any μ_p-measurable non-null set A, there is a cylinder C whose 99% is A, i.e. $\frac{\mu_p(C \cap A)}{\mu(C)} \ge 0.99$.

Proof. We prove (a), and (b) is similar.

We may assume WLOG that A is bounded by restricting to a large enough in which A has positive measure.

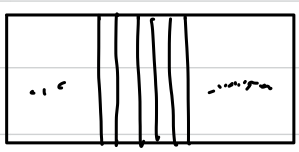
We already know that $\exists$ a finite disjoint union $\bigsqcup_{i \le n} B_i$ of boxes s.t. $\mu\left(A \Delta \bigsqcup_{i \le n} B_i\right) \le 1\% \mu\left(\bigsqcup_{i \le n} B_i\right)$.

In particular, $\frac{\mu\left(\bigsqcup_{i\le n} B_i \cap A\right)}{\mu\left(\bigsqcup_{i \le n} B_i\right)} \geq 0.99$. By percentage pigeon-hole principle, at least one B_i should be $\geq 99\%$ A because otherwise the percentage of A in $\bigsqcup_{i\le n} B_i$ would be $< 99\%$.

BTW: Instead of approximating by a finite union of boxes, we could have recalled that $\mu(A) = \mu^*(A) = \inf\{\bigvee_{n\in\mathbb{N}} B_n : n\in\mathbb{N}\}$ so $\exists$ cbl disjoint union $\bigsqcup_{n\in\mathbb{N}} B_n \supseteq A$ and s.t. A is 99% of $\bigsqcup_{n\in\mathbb{N}} B_n$. The percentage-pigeonhole still works due to cbl additivity. $\square$

Application to ergodic theory. Let E be an equivalence relation on a meas. space (X,μ). We say that E is *μ-ergodic* if each E-invariant (i.e. union of E-classes) μ-meas. set is either null or conull.

X



An action $\Gamma \curvearrowright X$ of a cbl grp Γ is called *μ-ergodic* if its orbit eq. rel. E_Γ is μ-ergodic. This is measure-theoretic transitivity, recalling that an action is transitive (i.e. has only one orbit) if and only if every invariant set is either $\emptyset$ or the whole space.

I'll show next time using the 99% lemma that $\mathbb{E}_{\mathbb{Q}}$ on $\mathbb{R}$ is λ-ergodic and you'll show in HW that $\mathbb{E}_0$ is $\mu_{\frac{1}{2}}$-ergodic.